A Mathematical Approach to Logic

Predicate calculus can be seen as the theory of a specific kind of lattices. It follows that lattice theory may provide a basis for the axiomatization of predicate calculus. In this paper we explore that track.
We not only want to construct the predicate calculus but we also want to use it for proving theorems concerning the calculus under construction. We will call the predicate calculus that we use 'boolean arithmetic'. We will use the notation and axiomatization from [DS90], albeit without the everywhere operator. We have no use for the everywhere operator since all our boolean expressions are scalar.
We will start with a complete lattice and, by adding postulates we will transform it into (something isomorphic to) the classical predicate calculus from [DS90]. En route, we will have the opportunity to study the constructive predicate calculus.
The reader is supposed to be familiar with both boolean arithmetic and lattice theory (lattice *calculus* in fact).

0 A Complete Lattice

In this paper we consider a set L, the elements of which we call 'specks'. On specks we have a partial order, $\leq$, and we postulate that

(0) $(L, \leq)$ is a complete lattice.

We will use $\sqcap$ and $\sqcup$ to denote infima and suprema respectively and we will use these both in quantifier format and in infix notation (a bit smaller in the latter case). We use $\top$ and $\bot$ to denote the largest and smallest speck respectively.

Thus, these operators and constants are characterized by the fact that we have, for all specks p and funtions f to L

(1) $p \leq (\sqcap x :: f.x) \;\equiv\; (\forall x :: p \leq f.x)$,

(2) $(\sqcup x :: f.x) \leq p \;\equiv\; (\forall x :: f.x \leq p)$,

(3) $\bot \leq p \leq \top$.

remark
We consider the infix operators to be just special cases of the quantifiers, i.e. quantification with a – small – finite range. Hence, anything that holds for a quantifier holds – mutatis mutandis – for the corresponding infix operator and we will, in general, not mention it separately for that special case. Note that making the translation often also involves replacing universal quantifications by conjunctions.
end of remark

As already noted above, we have no use for the everywhere operator in our boolean arithmetic since all our boolean expressions are scalar. We now (re-)introduce the everywhere operator as an operator from specks to booleans:

(4) $[\,p\,] \;\equiv\; \top \leq p$ or equivalently

$[\,p\,] \;\equiv\; \top = p$ for all specks p.

From (1) with $p = \top$ and the previous definition we immediately obtain

(5) $$[(\sqcap x :: f.x)] \equiv (\forall x :: [f.x]) .$$

Note that the everywhere operator destroys the symmetry between infima and suprema. We could, of course, re-establish the symmetry by introducing a corresponding 'nowhere operator', but we won't.

1 A Heyting Lattice

We are now going to add a postulate to the effect that $(L, \leq)$ is a 'complete Heyting lattice'. Such a lattice is the canonical model for the constructive and intuitionistic predicate logic. It should be noted however, that in these logics the formulae that may appear in the term of a quantification are subject to syntactic constraints. Consequently, the lattice structure of these logics is not necessarily as complete as ours.

There are many postulates that characterize a Heyting lattice and each may have its own merits. Since we do not want to select one at the expence of the others, we will give several of them and show their equivalence. Here they are:

(6) $\sqcap$ is universally $\sqcup$-junctive, i.e.

$p \sqcap (\sqcup x :: f.x) = (\sqcup x :: p \sqcap f.x)$ for all specks p and functions f to L.

(7) There exists an operator, $\rightleftharpoons$, from pairs of specks to specks, such that

$p \sqcap q = p \sqcap r \equiv p \leq q \rightleftharpoons r$ for all specks p, q and r .

(8) There exists an operator, $\rightarrow$, from pairs of specks to specks, such that

$p \sqcap q \leq r \equiv p \leq q \rightarrow r$ for all specks p, q and r .

Note that, even if we had restored the symmetry between infima and suprema at the end of the previous section, it would have been short lived, since the above postulates break the symmetry again.

We call $\rightleftharpoons$ 'the internal equivalence' and $\rightarrow$ 'the internal implication'. Note that for each of these operators we have that, if existent, it is unique.

The order of decreasing binding power for the operators that we have now in use is

$$(\sqcap \ \sqcup), \rightarrow, \rightleftharpoons, (\leq =)$$

where the paired operators have the same binding power.

Proof of the equivalence of (6), (7), and (8).

The equivalence of (6) and (8) is a standard ingredient of the theory of Galois connections, so we consider this dealt with.

We prove the equivalence of (8) and (7) by mutual implication.

(7)$\Leftarrow$(8) : Assuming (8), we observe for any p, q and r

$p \sqcap q = p \sqcap r$

$=$ { lattice calc. }

$p \sqcap q \leq r \ \wedge \ p \sqcap r \leq q$

$=$ { (8) twice }

$p \leq q \rightarrow r \ \wedge \ p \leq r \rightarrow q$

$=$ { lattice calc. }

$p \;\leq\; (q \rightarrow r) \sqcap (r \rightarrow q)$,

from which it follows that we can define the sought operator $\rightleftharpoons$ by

(9) $\quad q \rightleftharpoons r \;=\; (q \rightarrow r) \sqcap (r \rightarrow q)$ for all specks q and r .

(7)$\Rightarrow$(8) : Assuming (7), we observe for any p, q and r

$p \sqcap q \leq r$

$=$ { lattice calc. }

$p \sqcap q = p \sqcap q \sqcap r$

$=$ { (7) }

$p \;\leq\; q \rightleftharpoons q \sqcap r$,

from which it follows that we can define the sought operator $\rightarrow$ by

(10) $\quad q \rightarrow r \;=\; q \rightleftharpoons q \sqcap r$ for all specks q and r .

End of proof.

From now on we consider one (and hence all) of (6), (7), and (8) to be postulated. Note that, on account of the uniqueness of the operators $\rightarrow$ and $\rightleftharpoons$, we get the equalities (9) and (10) as bonus from the preceding proofs. We will now proceed by exploring the consequences of the postulates in the order in which these are given.

About (6) we will be short: as a special case of (6) we have that $\sqcap$ distributes over $\sqcup$. It is a standard ingredient of lattice theory that we now have the reverse distribution as well, i.e.

(11) the binary operators $\sqcap$ and $\sqcup$ distribute over each other.

So much for (6). We will dwell considerably longer on the consequences of (7).

Instantiating (7) with $p := \top$, we obtain that

(12) $\quad q = r \;\equiv\; [\, q \rightleftharpoons r \,]$.

Since equality is symmetric, the left hand side of (7) is symmetric in q and r. It follows that the same holds for the right hand side:

$p \leq q \rightleftharpoons r \;\equiv\; p \leq r \rightleftharpoons q$ for all specks p, q and r .

Which observation allows to conclude that

(13) $\rightleftharpoons$ is symmetric.

Instantiating (7) with $r := \top$ we find (after rewriting the resulting left hand side):

$p \leq q \;\equiv\; p \leq q \rightleftharpoons \top$ for all specks p and q .

Which observation allows to conclude that

(14) $\top$ is the neutral element of $\rightleftharpoons$.

Instantiating (7) with $p := q \rightleftharpoons r$ we obtain:

(15) $\quad (q \rightleftharpoons r) \sqcap q \;=\; (q \rightleftharpoons r) \sqcap r$.

These were easy. We proceed with a batch of theorems that require more extensive proofs.

(16) $\quad (\sqcap x :: f.x \rightleftharpoons g.x) \;\leq\; (\sqcap x :: f.x) \rightleftharpoons (\sqcap x :: g.x)$.

(17) $\quad (\sqcap x :: f.x \rightleftharpoons g.x) \;\leq\; (\sqcup x :: f.x) \rightleftharpoons (\sqcup x :: g.x)$.

(18) $\quad p \rightleftharpoons q \;\leq\; p \sqcap r \rightleftharpoons q \sqcap r$.

(19) $\quad p \rightleftharpoons q \;\leq\; p \sqcup r \rightleftharpoons q \sqcup r$.

(20) $\quad p \rightleftharpoons q \;\leq\; (p \rightleftharpoons r) \rightleftharpoons (q \rightleftharpoons r)$.

proofs

Ad (16) : We observe

$(\sqcap x :: f.x \rightleftharpoons g.x) \sqcap (\sqcap x :: f.x)$

= { lattice calc. }

$(\sqcap x :: (f.x \rightleftharpoons g.x) \sqcap f.x)$

= { (15) }

$(\sqcap x :: (f.x \rightleftharpoons g.x) \sqcap g.x)$

= { lattice calc. }

$(\sqcap x :: f.x \rightleftharpoons g.x) \sqcap (\sqcap x :: g.x)$,

and from that (16) follows by (7).

Ad (17) : We observe for any p

$p \;\leq\; (\sqcup x :: f.x) \rightleftharpoons (\sqcup x :: g.x)$

= { (7) }

$p \sqcap (\sqcup x :: f.x) \;=\; p \sqcap (\sqcup x :: g.x)$

= { (6) }

$(\sqcup x :: p \sqcap f.x) \;=\; (\sqcup x :: p \sqcap g.x)$

$\Leftarrow$ { lattice calc. }

$(\forall x :: p \sqcap f.x \;=\; p \sqcap g.x)$

= { (7) }

$(\forall x :: p \;\leq\; f.x \rightleftharpoons g.x)$

= { lattice calc. }

$p \;\leq\; (\sqcap x :: f.x \rightleftharpoons g.x)$,

from which (17) follows.

Ad (18) and (19) : These are special instances of (16) and (17) respectively (note that, on account of (7), $r \rightleftharpoons r \;=\; \top$).

Ad (20) : By (7), we have to prove

$(p \rightleftharpoons q) \sqcap (p \rightleftharpoons r) \;=\; (p \rightleftharpoons q) \sqcap (q \rightleftharpoons r)$.

To this end we observe for any s :

$s \;\leq\; (p \rightleftharpoons q) \sqcap (p \rightleftharpoons r)$

= { lattice calc. and (7) }

$s \sqcap p = s \sqcap q \;\;\wedge\;\; s \sqcap p = s \sqcap r$

= { Leibniz }

$s \sqcap p = s \sqcap q \;\;\wedge\;\; s \sqcap q = s \sqcap r$

= { (7) and lattice calc. }

$$s \leq (p \rightleftharpoons q) \sqcap (q \rightleftharpoons r) \ .$$

end of proofs

The last three theorems all have the same shape, viz.

$p \rightleftharpoons q \leq f.p \rightleftharpoons f.q$ for all specks p and q,

which expresses a property of the function f. We call functions that have this property 'internal functions', i.e.

(21) f is internal $\equiv$

$p \rightleftharpoons q \leq f.p \rightleftharpoons f.q$ for all specks p and q,

or equivalently

(22) f is internal $\equiv$

$(p \rightleftharpoons q) \sqcap f.p = (p \rightleftharpoons q) \sqcap f.q$ for all specks p and q.

We now turn our attention to functions with more than one argument. For succinctness we will treat binary operators and we leave it to the reader to make the generalization to any number.

For a binary operator, $*$, we define

(23) $*$ is internal in its first argument $\equiv$

$p \rightleftharpoons q \leq p{*}r \rightleftharpoons q{*}r$ for all specks p, q, and r,

and similarly for 'being internal in the second argument'. Operators that are internal in all their arguments, we will call simply 'internal'. Note that for a *symmetric* binary operator, 'being internal in one of the aguments' is the same as 'being internal (in both arguments)'.
Since $\sqcap$, $\sqcup$, and $\rightleftharpoons$ are symmetric and, by (18), (19), and (20), internal in their first argument, we can now summarize these theorems by
(24) $\sqcap$, $\sqcup$, and $\rightleftharpoons$ are internal.

For binary $*$, the property of being internal can also be characterized in one go (instead of seperately for each argument):

(25) $*$ is internal $\equiv$

$(p \rightleftharpoons q) \sqcap (x \rightleftharpoons y) \leq p{*}x \rightleftharpoons q{*}y$ for all specks p, q, x and y.

proof
By mutual implication.
$\Leftarrow$: From the special cases $x = y$ and $p = q$ respectively.
$\Rightarrow$: We observe for any p, q, x and y

$$
\begin{array}{ll}
 & p{*}x \rightleftharpoons q{*}y \\
\geq & \quad \{ \text{ lattice calc. } \} \\
 & (p{*}x \rightleftharpoons q{*}x) \sqcap (p{*}x \rightleftharpoons q{*}y) \\
= & \quad \{ \rightleftharpoons \text{ is internal, (22) with } f := (\boldsymbol{\lambda}\, z :: z \rightleftharpoons q{*}y) \ \} \\
 & (p{*}x \rightleftharpoons q{*}x) \sqcap (q{*}x \rightleftharpoons q{*}y) \\
\geq & \quad \{ * \text{ is internal, monotonicity of } \sqcap \ \} \\
 & (p \rightleftharpoons q) \sqcap (x \rightleftharpoons y) \ .
\end{array}
$$

end of proof

We generalize (25) to a definition for a function of an arbitrary number of arguments and replacing the infix $\sqcap$ by the quantifier $\sqcap$. Thus, (16) and (17) can now be read as stating that

(26) the quantifiers $\sqcap$ and $\sqcup$ are internal.

The property of 'being internal' is preserved under under all manners of combination:

(27) If f and g are internal functions and $*$ is internal in both its arguments, then is $f.p * g.p$ an internal function of p.

proof

$f.p * g.p \rightleftharpoons f.q * g.q$

$\geq$ { $*$ is internal in both arguments, (25) }

$(f.p \rightleftharpoons f.q) \sqcap (g.p \rightleftharpoons g.q)$

$\geq$ { f and g are internal }

$p \rightleftharpoons q$.

end of proof

On account of (27), internal functions and operators can be freely combined to yield new internal functions. Since constant functions and the identity function are trivially internal, we obtain by induction on the syntax:

(28) An expression build from constants, variables, and internal function and operators, is an internal function of each of its variables.

Now, as a demonstration of the utility of the concept we show

(29) $p \rightleftharpoons p \sqcap q \;=\; p \sqcup q \rightleftharpoons q$.

proof

$p \rightleftharpoons p \sqcap q$

$\leq$ { $\sqcup$ is internal }

$p \sqcup q \rightleftharpoons (p \sqcap q) \sqcup q$

$=$ { absorption }

$p \sqcup q \rightleftharpoons q$

$\leq$ { $\sqcap$ is internal }

$p \sqcap (p \sqcup q) \rightleftharpoons p \sqcap q$

$=$ { absorption }

$p \rightleftharpoons p \sqcap q$

from which equality follows by anti-symmetry
end of proof

The primary reason for including this intermezzo on internal functions is that it is the lattice theoretical version of the 'punctuality theory' of [DS90]. This punctuality theory is developed for the predicate calculus, and we thought it nice to show that the concept makes perfect sense in the – far more general – context of Heyting lattices.
Now that we have all this, we may as well use it. However, we need to do something about our proof format first. In particular, in order to meet a proof obligation of the form $p \leq q \rightleftharpoons r$ we have to show that $p \sqcap q \;=\; p \sqcap r$. If we do so by transforming one side

into the other through a series of value preserving steps, the chances are good that we will see $p \sqcap \ldots$ at the start of every line. Since – possibly numerous – repetitions of such a – possibly lengthy – subexpression can be very annoying, we allow it be left implicit. We call such a calculation a 'calculation under proviso', and a proof of $p \le q \rightleftharpoons r$ could now read something like:

Under the proviso p we observe:

$$
\begin{array}{ll}
 & q \\
= & \{ \text{ hint why } q = x \ \} \\
 & x \\
= & \{ \text{ hint why } p \sqcap x = p \sqcap y \text{ or equivalently } p \le x \rightleftharpoons y \ \} \\
 & y \\
= & \{ \text{ hint why } y = r \ \} \\
 & r \ .
\end{array}
$$

Please take proper note of the special case that $r = \top$. Since $\top$ is the neutral element of the internal equivalence, it would not appear explicitly in the proof obligation, but it would pop up in the last line of the proof.

Note further that in proofs under the proviso $p \rightleftharpoons q$, we can now have steps like

$$
\begin{array}{ll}
 & f.p \\
= & \{ \text{ proviso, } f \text{ is internal } \} \\
 & f.q \ ,
\end{array}
$$

and that, again, in the special case that $q = \top$, the expression $\top$ is not explicitly present in the proviso, but may appear in the proof.

To illustrate how this works we give two proofs of

(30) $\quad (p \rightleftharpoons q) \sqcap (q \rightleftharpoons r) \ \le \ p \rightleftharpoons r \ .$

proofs

Under the proviso $(p \rightleftharpoons q) \sqcap (q \rightleftharpoons r)$ we observe

$$
\begin{array}{ll}
 & p \rightleftharpoons r \\
= & \{ \text{ proviso, } \rightleftharpoons \text{ is internal } \} \\
 & q \rightleftharpoons q \\
= & \{ \ (12) \ \} \\
 & \top \ .
\end{array}
$$

or (under the same proviso)

$$
\begin{array}{ll}
 & p \\
= & \{ \text{ proviso } \} \\
 & q \\
= & \{ \text{ proviso } \} \\
 & r \ .
\end{array}
$$

end of proof

The time has come to have a look at the last postulate that introduces the internal implication. We have two angles on this operator: the equivalence of (8) and the equality of (10). We begin with some consequences of the first.

Instantiating (8) with $p := \top$ we obtain :

(31) $\quad q \leq r \;\equiv\; [\, q \rightarrow r \,]$.

and, using (31) to reformulate (3), we get:

(32) $\quad [\, \bot \rightarrow p \,]$ and $[\, p \rightarrow \top \,]$.

Since (8) expresses a Galois connection, we have that $\rightarrow$ is universally $\sqcap$-junctive in its second argument:

(33) $\quad p \rightarrow (\sqcap x :: f.x) \;=\; (\sqcap x :: p \rightarrow f.x)$.

Since $\sqcap$ is symmetric, the left hand side of (8) is symmetric in p and q. It follows that the same holds for the right hand side:

(34) $\quad p \leq q \rightarrow r \;\equiv\; q \leq p \rightarrow r$ for all specks p, q and r .

Now, replacing in (34) the left hand side by $q \rightarrow r \geq p$, we recognize another Galois connection (between $(L, \leq)$ and its dual). So, we get the following distributive property of $\rightarrow$ in the first argument:

(35) $\quad (\sqcup x :: f.x) \rightarrow p \;=\; (\sqcap x :: f.x \rightarrow p)$.

(Note that the suprema of $(L, \leq)$ are the infima of its dual and vice versa).

Immediate consequences of (33) and (35) are

(36) $\quad \rightarrow$ is monotonic in its second argument and anti-monotonic in its first argument.

We have already investigated the internal equivalence quite extensively so we may expect some theorems to be cheaply obtainable from (10). We first observe that (10) expresses $\rightarrow$ in terms of the internal operators $\sqcap$ and $\rightleftharpoons$. Thus we have by (28) that

(37) $\quad \rightarrow$ is internal.

Since $\top$ is the neutral element of both $\sqcap$ and $\rightleftharpoons$ we get from (10) with $q := \top$ that

(38) $\quad \top$ is a left-neutral element of $\rightarrow$.

We also easily obtain the following

(39) 'complement rules':

(i) $\quad p \rightarrow (p \sqcap q) \;=\; p \rightarrow q$,

(ii) $\quad p \sqcap (p \rightarrow q) \;=\; p \sqcap q$.

(Note that we added some redundant brackets to the first of these to enhance the symmetry)

proof

We prove (i) by observing

$\quad p \rightarrow (p \sqcap q) \;=\; p \rightarrow q$

$= \quad$ { (10) (twice) }

$\quad p \rightleftharpoons p \sqcap (p \sqcap q) \;=\; p \rightleftharpoons p \sqcap q$

$= \quad$ { associativity and idempotence of $\sqcap$ }

$\quad$ *true* .

We prove (ii) by observing under proviso p

$$
\begin{array}{ll}
 & p \to q \\
= & \quad \{ \text{ proviso, } \to \text{ is internal – by (37) – } \} \\
 & \top \to q \\
= & \quad \{ \top \text{ neutral –by (38) – } \} \\
 & q
\end{array}
$$

end of proof

And now we're ready for the beautiful

(40) '$\to$ over $\rightleftharpoons$' :

(i) $p \sqcap q \rightleftharpoons p \sqcap r \ = \ p \to (q \rightleftharpoons r)$,

(ii) $p \to q \rightleftharpoons p \to r \ = \ p \to (q \rightleftharpoons r)$.

proof

We prove (i) by observing for any s

$$
\begin{array}{ll}
 & s \le p \to (q \rightleftharpoons r) \\
= & \quad \{ (8) \} \\
 & s \sqcap p \le q \rightleftharpoons r \\
= & \quad \{ (7) \} \\
 & s \sqcap p \sqcap q \ = \ s \sqcap p \sqcap r \\
= & \quad \{ (7) \} \\
 & s \le p \sqcap q \rightleftharpoons p \sqcap r \ .
\end{array}
$$

Now, since the right hand sides of (i) and (ii) are equal, we can prove (ii) by proving the equality of the left hand sides.

$$
\begin{array}{ll}
 & p \sqcap q \rightleftharpoons p \sqcap r \\
\le & \quad \{ \to \text{ is internal } \} \\
 & p \to (p \sqcap q) \rightleftharpoons p \to (p \sqcap r) \\
= & \quad \{ \text{ complement rule (39.i) } \} \\
 & p \to q \rightleftharpoons p \to r \\
\le & \quad \{ \sqcap \text{ is internal } \} \\
 & p \sqcap (p \to q) \rightleftharpoons p \sqcap (p \to r) \\
= & \quad \{ \text{ complement rule (39.ii) } \} \\
 & p \sqcap q \rightleftharpoons p \sqcap r \ ,
\end{array}
$$

And this proves equality by anti-symmetry.

end of proof

Note the similarity between (7) and (40.i). Note further that (40.ii) expresses that the internal implication distributes from the left over the internal equivalence.
Instantiating (40) with $r := q \sqcap r$ and rewriting the results using (10) (and (33) in the second case) yields

(41) '$\to$ over $\to$' :

(i) $\quad p \sqcap q \to r \;=\; p \to (q \to r)$,

(ii) $\quad (p \to q) \to (p \to r) \;=\; p \to (q \to r)$.

Note, in this case, the similarity between (8) and (41.i) and the fact that (41.ii) expresses that the internal implication distributes from the left over itself.

It is time to introduce our next operator. Consider, for each of the binary lattice operators, the unary operator obtained by fixing one of the arguments to either $\top$ or $\bot$. One easily verifies that, with the exception of the functions that map p to $p \rightleftharpoons \bot$ and $p \to \bot$ respectively, we obtain either a constant function or the identity. Furthermore, it follows from (10) that our two exceptions are in fact equal. Thus we define the operator $\bullet$ by

(42) $\quad \bullet p \;=\; p \rightleftharpoons \bot \quad$ or equivalently

$\quad \bullet p \;=\; p \to \bot$.

The operator $\bullet$ is called 'the pseudo-complement' and we give it a binding power which is higher than that of the binary lattice operators.

The properties of the pseudo-complement of a Heyting lattice are considerably less pleasant than those of the full complement of a Boolean lattice. Boolean lattices are the subject of the next section and preparation of the ground for that section is our only reason for including the pseudo-complement in our investigation here.

Some facts that are good to know about the pseudo-complement are:

(43) $\quad \bullet\bot \;=\; \top \quad$ and $\quad \bullet\top \;=\; \bot$.

(44) $\quad \bullet(\sqcup x :: f.x) \;=\; (\sqcap x :: \bullet f.x)$.

(45) $\quad \bullet$ is internal and anti-monotonic.

(46) $\quad \bullet(p \sqcap q) \;=\; p \to \bullet q$.

(47) $\quad p \to \bullet q \;=\; q \to \bullet p$.

(48) $\quad p \sqcap q = \bot \;\equiv\; p \le \bullet q$.

(49) $\quad p \le \bullet q \;\equiv\; q \le \bullet p$.

(50) $\quad \bullet q \sqcap q \;=\; \bot$.

(51) $\quad \bullet q \rightleftharpoons q \;=\; \bot$.

(52) $\quad p \le \bullet\bullet p$.

(53) $\quad \bullet\bullet\bullet p \;=\; \bullet p$.

(54) $\quad \bullet(q \rightleftharpoons \bullet r) \;=\; \bullet q \rightleftharpoons \bullet r$.

(55) $\quad \bullet p \rightleftharpoons (q \rightleftharpoons \bullet r) \;=\; (\bullet p \rightleftharpoons q) \rightleftharpoons \bullet r$.

proofs

Ad (43) : From (12) and (14).

Ad (44) : Instantiate (35) with $p := \bot$.

Ad (45) : Immediate, since $\to$ is internal and anti-monotonic in its first argument.

Ad (46) : Instantiate either (40.i) or (41.i) with $r := \bot$.

Ad (47) : From (46), since the left-hand side is symmetric in p and q.

Ad (48) : Apply the everywhere operator to both sides of (46) and use (12) and (31).

Ad (49) : From (48), since the left-hand side is symmetric in p and q. Alternatively, apply the everywhere operator to both sides of (47).

Ad (50) : Instantiate (48) with $p := \bullet q$.

Ad (51) :

$$
\begin{array}{ll}
 & \bullet q \rightleftharpoons q \\
= & \quad \{ \ (9) \ \} \\
 & (q \rightarrow \bullet q) \sqcap (\bullet q \rightarrow q) \\
= & \quad \{ \ (46), \text{idempotence of } \sqcap \ \} \\
 & \bullet q \sqcap (\bullet q \rightarrow q) \\
= & \quad \{ \ \text{complement rule (39.ii)} \ \} \\
 & \bullet q \sqcap q \\
= & \quad \{ \ (50) \ \} \\
 & \bot \ .
\end{array}
$$

Ad (52) : Instantiate (49) with $q := \bullet p$.

Ad (53) :

$$
\begin{array}{ll}
 & \bullet\bullet\bullet p \ = \ \bullet p \\
= & \quad \{ \ \geq \text{ holds on account of (52) with } p := \bullet p \ \} \\
 & \bullet\bullet\bullet p \leq \bullet p \\
\Leftarrow & \quad \{ \ \bullet \text{ is anti-monotonic by (45)} \ \} \\
 & p \leq \bullet\bullet p \\
= & \quad \{ \ (52) \ \} \\
 & \mathit{true} \ .
\end{array}
$$

Before we prove the last two theorems, note that (54) is in fact a special case of (55) (with $p := \top$). The reason for proving (54) separately is that we use it in the proof of (55).
Note further that the theorems express – restricted forms of – transitivity. Proofs of transitivity tend to be messy and the following two are no exceptions.

Ad (54) : By ping-pong argument.

ping

$$
\begin{array}{ll}
 & \bullet(q \rightleftharpoons \bullet r) \ \leq \ \bullet q \rightleftharpoons \bullet r \\
= & \quad \{ \ (7) \ \} \\
 & \bullet(q \rightleftharpoons \bullet r) \sqcap \bullet q \ = \ \bullet(q \rightleftharpoons \bullet r) \sqcap \bullet r \\
= & \quad \{ \ \bullet q \ = \ q \rightleftharpoons \bot \text{ and } \bullet r \ = \ \bullet r \rightleftharpoons \top, \text{ both } \bullet \text{ and } \rightleftharpoons \text{ are internal} \ \} \\
 & \bullet(\bot \rightleftharpoons \bullet r) \sqcap \bullet q \ = \ \bullet(q \rightleftharpoons \top) \sqcap \bullet r \\
= & \quad \{ \ \text{definition of } \bullet, \text{ neutrality of } \top \ \} \\
 & \bullet\bullet\bullet r \sqcap \bullet q \ = \ \bullet q \sqcap \bullet r \\
= & \quad \{ \ \text{symmetry of } \sqcap \text{ and (53)} \ \} \\
 & \mathit{true} \ .
\end{array}
$$

pong Under the proviso $\bullet q \rightleftharpoons \bullet r$ we observe

$$q \rightleftharpoons \bullet r$$

$=$ { proviso, $\rightleftharpoons$ is internal }

$q \rightleftharpoons \bullet q$

$=$ { (51) }

$\bot$.

Since $(q \rightleftharpoons \bullet r) \rightleftharpoons \bot \ = \ \bullet(q \rightleftharpoons \bullet r)$ by definition, this proves 'pong'.

Ad (55) : By ping-pong argument. For reasons of symmetry, just 'ping' suffices.

$\bullet p \rightleftharpoons (q \rightleftharpoons \bullet r)$

$=$ { (9) }

$(\bullet p \to (q \rightleftharpoons \bullet r)) \sqcap ((q \rightleftharpoons \bullet r) \to \bullet p)$

$=$ { (47), (54) }

$(\bullet p \to (q \rightleftharpoons \bullet r)) \sqcap (p \to (\bullet q \rightleftharpoons \bullet r))$

$=$ { $\to$ over $\rightleftharpoons$ }

$(\bullet p \to q \rightleftharpoons \bullet p \to \bullet r) \sqcap (p \to \bullet q \rightleftharpoons p \to \bullet r)$

$=$ { (47) (thrice) }

$(\bullet p \to q \rightleftharpoons r \to \bullet\bullet p) \sqcap (q \to \bullet p \rightleftharpoons r \to \bullet p)$

$\leq$ { $\sqcap$ is internal }

$(\bullet p \to q) \sqcap (q \to \bullet p) \ \rightleftharpoons \ (r \to \bullet\bullet p) \sqcap (r \to \bullet p)$

$=$ { (9), $\to$ over $\sqcap$ }

$(\bullet p \rightleftharpoons q) \rightleftharpoons r \to (\bullet\bullet p \sqcap \bullet p)$

$=$ { $\bullet\bullet p \sqcap \bullet p \ = \ \bot$ by (50), definition of $\bullet$ }

$(\bullet p \rightleftharpoons q) \rightleftharpoons \bullet r$.

end of proofs

2 A Boolean Lattice

In the previous section we have explored the consequences of adding a single postulate to a complete lattice (allowing a choice from several equivalent ones). In this section we add a second postulate. The lattice structure that we so obtain is 'a complete Boolean lattice' and it is the canonical model for the classical predicate logic. As before, there are several alternatives and, since we do not want to deny the reader the freedom of choice, we give them all and show their equivalence.

(56) $\bullet q \to r \ = \ q \sqcup r$.

(57) $\sqcup$ distributes over $\rightleftharpoons$.

(58) $q \to r \ = \ \bullet q \sqcup r$.

(59) $p \sqcap q \leq r \ \ \equiv \ \ p \leq \bullet q \sqcup r$.

(60) $[\, \bullet q \sqcup q \,]$.

(61) $\bullet$ is an involution.

(62) $p \rightleftharpoons (q \rightleftharpoons r) \ = \ (p \rightleftharpoons q) \rightleftharpoons r$ i.e. $\rightleftharpoons$ is associative.

(63) $\bullet(q \rightleftharpoons r) \ = \ \bullet q \rightleftharpoons r$.

remark
A unary operator for which (59) holds is called 'a full complement'. We will show the equivalence of (56) through (63) in the context of a Heyting lattice. However, it should be noted that the existence of a full complement implies the existence of an internal implication (we could define it by (58)). Thus, in contrast to the others, (59) also implies all of the previous section.
end of remark

Proof of the equivalence of (56) through (63)

We prove the equivalence by means of the following cycle of implications:

$$(56) \Rightarrow (57) \Rightarrow (58) \Rightarrow (59) \Rightarrow (60) \Rightarrow (61) \Rightarrow (62) \Rightarrow (63) \Rightarrow (61) \Rightarrow (56) \ .$$

(Note that our 'cycle' is actually a figure 8)

(56) ⇒ (57) : Instantiate (40.ii) with $p := \bullet p$ and rewrite using (56).

(57) ⇒ (58) :

$$\begin{aligned}
& \bullet q \sqcup r \\
= \quad & \{ \text{ definition of } \bullet \ \} \\
& (q \rightleftharpoons \bot) \sqcup r \\
= \quad & \{ \ (57), \bot \text{ is the neutral element of } \sqcup \ \} \\
& q \sqcup r \rightleftharpoons r \\
= \quad & \{ \ (29) \ \} \\
& q \rightleftharpoons q \sqcap r \\
= \quad & \{ \ (10) \ \} \\
& q \rightarrow r \ .
\end{aligned}$$

(58) ⇒ (59) : Rewrite (8) using (58).

(59) ⇒ (60) : Instantiate (59) with $p, r := \top, q$.

(60) ⇒ (61) :

$$\begin{aligned}
& \bullet\bullet q \\
= \quad & \{ \ (60) \ \} \\
& \bullet\bullet q \sqcap (\bullet q \sqcup q) \\
= \quad & \{ \ \sqcap \text{ over } \sqcup \ \} \\
& (\bullet\bullet q \sqcap \bullet q) \sqcup (\bullet\bullet q \sqcap q) \\
= \quad & \{ \ \bullet\bullet q \sqcap \bullet q = \bot \text{ by } (50) \ \} \\
& (\bullet\bullet q \sqcap q) \\
= \quad & \{ \ q \leq \bullet\bullet q \text{ by } (52) \ \} \\
& q \ .
\end{aligned}$$

(61) ⇒ (62) : Instantiate (55) with $p, r := \bullet p, \bullet r$ and rewrite using (61).

(62) ⇒ (63) : Instantiate (62) with $p := \bot$.

(63) ⇒ (61) : Instantiate (63) with $q := \bot$.

(61) ⇒ (56) :

$$\begin{aligned}
& \bullet q \rightarrow r \\
= \quad & \{ \ (61) \ \} \\
& \bullet q \rightarrow \bullet\bullet r
\end{aligned}$$

$$
\begin{array}{ll}
= & \{ (46) \} \\
& \bullet(\bullet q \sqcap \bullet r) \\
= & \{ (44) \} \\
& \bullet\bullet(q \sqcup r) \\
= & \{ (61) \} \\
& q \sqcup r \ .
\end{array}
$$

end of proofs

From now on we consider one – and hence all – of the assertions (56) through (63) to be postulated.
Apart from giving the reader the freedom of choice, there is another advantage of having established the equivalences above. When proving something for Boolean lattices, the question naturally arises whether it holds for Heyting lattices as well. Since there are Heyting lattices that are not Boolean – e.g. the real interval $[0\,..\,1]$ with the standard ordering on real numbers –, we now know that none of the assertions (56) through (63) hold for Heyting lattices in general.

One reason for writing this note was to verify that the postulates for the predicate calculus of [DS90] describe exactly a complete Boolean lattice. (We used, for the lattice opertors, very thinly disguised logical operators, so we trust that the reader can make the match.)

One direction of the verification is easy. Defining the relation $\leq$ on predicates by

$$P \leq Q \ \equiv \ [\, P \Rightarrow Q \,] \ ,$$

we find that this is an ordering relation on predicates. Since universal and existential quantification denote arbitrary infima and suprema respectively, we have indeed a complete lattice. Finally, (59) is satisfied on account of 'the shunting theorem', so this complete lattice is Boolean (see the remark below (63)).

For the reverse direction, we have to check that all the postulates of [DS90] hold for a complete Boolean lattice. Since the number of postulates to check is large, this is a lot of work. We have, however, already almost everything we need. First recall that (12) states that

$$[\, p \rightleftharpoons q \,] \ \equiv \ p = q \ ,$$

which justifies the version of the rule of Leibniz that is used in [DS90]. We will, henceforth, generally use $[\, .. \rightleftharpoons .. \,]$ to express equality.
Secondly, recall that (62) states that

$\rightleftharpoons$ is associative,

which we will use to omit redundant parentheses.
Part of the postulates of [DS90] consist of typing: for each logical operator it is postulated that the set of boolean scalars is closed under it. The reader is invited to check that, for the lattice, we have:

The set $\{\top, \bot\}$ is closed under all lattice operators (including the quantifiers).

And now we give an almost exhaustive list of (the lattice versions of) the remaining postulates of [DS90]. Each of them is followed by a hint which should suffice for its verification.

$[\, p \rightleftharpoons q \rightleftharpoons q \rightleftharpoons p \,]$	(13)
$[\, p \rightleftharpoons \top \rightleftharpoons p \,]$	(14)
$\sqcup$ is symmetric, associative, and idempotent.	lattice calc.
$\sqcup$ distributes over $\rightleftharpoons$.	(57)
$[\, p \sqcap q \rightleftharpoons p \rightleftharpoons q \rightleftharpoons p \sqcup q \,]$	(29)
$[\, p \to q \rightleftharpoons p \sqcup q \rightleftharpoons q \,]$	(10), (29)
$\bot = \bullet\top$	(43)
$[\, \bullet(p \rightleftharpoons q) \rightleftharpoons \bullet p \rightleftharpoons q \,]$	(63)
$[\, p \sqcup \bullet p \,]$	(60)
$[\, p \sqcap q \,] \;\equiv\; [\, p \,] \wedge [\, q \,]$	(5)
$[\, p \sqcup (\sqcap x :: f.x) \rightleftharpoons (\sqcap x :: p \sqcup f.x) \,]$	(56), (33)
$[\, (\sqcap x :: f.x \sqcap g.x) \rightleftharpoons (\sqcap x :: f.x) \sqcap (\sqcap x :: g.x) \,]$	lattice calc.
$[\, (\sqcap x :: (\sqcap y :: f.(x,y)) \rightleftharpoons (\sqcap y :: (\sqcap x :: f.(x,y)) \,]$	lattice calc.
$[\, (\sqcap x :: f.x) \,] \;\equiv\; (\forall x :: [\, f.x \,])$	(5)
$[\, (\sqcap x : x = y : f.x) \rightleftharpoons f.y \,]$	lattice calc.
$f = g \;\Rightarrow\; [\, (\sqcap x :: f.x) \rightleftharpoons (\sqcap x :: g.x) \,]$	Leibniz
$[\, (\sqcup x :: f.x) \rightleftharpoons \bullet(\sqcap x :: \bullet f.x) \,]$	(61), (44)

There are two postulates from [DS90] that we have not adressed yet because they present a typing conflict. These postulates are

the everywhere operator is idempotent.

$b \vee [\, p \,] \;\equiv\; [\, b \vee p \,]$ for all booleans b and all predicates p.

Our problem is that we defined the everywhere operator as function from specks to booleans and that the latter are not, necessarily, included in the lattice. This is easily remedied by postulating that

$$\mathit{true} = \top$$

$$\mathit{false} = \bot \,.$$

The reader may verify that now each of the logical operators is the restriction of the corresponding lattice operator to the extrema of the lattice and that we indeed have that

the everywhere operator is idempotent, and

$b \sqcup [\, p \,] \rightleftharpoons [\, b \sqcup p \,]$ for all booleans b and all specks p.

The second reason for writing this note was that, some time ago, the realization dawned upon me that, stripping from the predicate calculus of [DS90] all 'non-constructive' elements, left me with a slightly more general lattice structure. This insight gave me a handle on the constructive logic that was very welcome because I have more affinity with algebra than with deduction systems. Since then, I learned from [TD88] that the lattice structure of constructive logic is called a Heyting lattice.

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